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LETTER TO THE EDITOR

A new conservation law constructed without using either Lagrangians or Hamiltonians

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Abstract. A new conservation theorem is derived. The conserved quantity is constructed in terms of a symmetry transformation vector of the equations of motion only, without using either Lagrangian or Hamiltonian structures (which may even fail to exist for the equations at hand). One example and implications of the theorem on the structure of point symmetry transformations are presented.

Conservation theorems are extremely useful tools in theoretical physics. Most of them are constructed by using either the Lagrangian or the Hamiltonian structures of the equations of motion. The Noether theorem associates one conserved quantity to each Lagrangian symmetry [1] while the Poisson theorem requires two constants of motion and the symplectic structure of the Hamiltonian formalism (Poisson bracket) to produce a third conserved quantity [2].

Other theorems rely on the previous knowledge of a symmetry transformation vector and a constant of motion, to get a new constant of motion, for instance [3]. Non-Noetherian conservation laws give rise to several constants of motion for a given symmetry transformation using the Langrangian structure of the equations of motion [3].

In this letter we present a new conservation theorem constructed in terms of a symmetry transformation vector of the equations of motion only. Neither a Lagrangian nor a Hamiltonian structure of the differential system is invoked to get the conservation law. No knowledge of a previously found constant of the motion is needed. In this sense, the theorem appears to differ drastically from the other conservation laws known up to now.

In what follows, we prove two versions of the theorem, we show that point symmetry transformations have a definite functional structure as a consequence of this new conservation law, and we present an illustration of the theorem.

Let us now state the theorem. Consider a set of second-order differential equations

$$\ddot{q}^i - F^i(q^j, \dot{q}^j, t) = 0 \qquad i, j = 1, \dots, n \tag{1}$$

where the force F^i satisfies (in some coordinate system)

$$\frac{\partial F^i}{\partial \dot{q}^i} = 0. \tag{2}$$

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If ξ^i

$$\xi^i = \xi^i(q^j, \dot{q}^j, t) \tag{3}$$

is a symmetry vector for equation (1), that is, if ξ^i satisfies

$$\frac{\bar{d}}{dt}\left(\frac{\bar{d}}{dt}\xi^i\right) - \frac{\partial F^i}{\partial q^j}\xi^j - \frac{\partial F^i}{\partial \dot{q}^j}\frac{\bar{d}}{dt}\xi^j = 0 \tag{4}$$

with

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + F^i\frac{\partial}{\partial \dot{q}^i} \tag{5}$$

then

$$I = \frac{\partial \xi^i}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i}\left(\frac{\bar{d}}{dt}\xi^i\right) \tag{6}$$

is a conserved quantity for equation (1). Before proving the theorem, a few remarks are in order. Condition (2) is only one very mild restriction on the force, which is satisfied by all velocity-independent forces (even if no potential for them exists). This condition is relaxed in the second version of the theorem. Lorentz-like forces also satisfy condition (2) even if the 'electromagnetic' fields are not derivable from potentials. The symmetry vector ξ^i is defined so that the infinitesimal transformation

$$q'^i = q^i + \varepsilon\xi^i \tag{7}$$

maps solutions q^j of equation (1) in solutions q'^i of the same equation (up to ε^2 terms). For details see [4] for instance. In the case of Lagrangian theories it may be shown that all symmetry transformations which satisfy the Noether theorem obey equation (4). (See, for instance, [5]).

Now for the proof. Consider

$$\frac{\bar{d}}{dt}I = \frac{\bar{d}}{dt}\frac{\partial \xi^i}{\partial q^i} + \frac{\bar{d}}{dt}\frac{\partial}{\partial \dot{q}^i}\frac{\bar{d}}{dt}\xi^i. \tag{8}$$

It is straightforward to show that for any function $A = A(\dot{q}^j, \dot{q}^j, t)$

$$\frac{\bar{d}}{dt}\frac{\partial}{\partial q^i}A - \frac{\partial}{\partial q^i}\frac{\bar{d}}{dt}A = -\frac{\partial F^j}{\partial q^i}\frac{\partial A}{\partial \dot{q}^j} \tag{9}$$

and

$$\frac{\bar{d}}{dt}\frac{\partial}{\partial \dot{q}^i}A - \frac{\partial}{\partial \dot{q}^i}\frac{\bar{d}}{dt}A = -\frac{\partial A}{\partial q^i} - \frac{\partial F^j}{\partial \dot{q}^i}\frac{\partial A}{\partial \dot{q}^j}. \tag{10}$$

Therefore, one may rewrite (8) as

$$\frac{\bar{d}I}{dt} = -\frac{\partial F^j}{\partial q^i}\frac{\partial \xi^i}{\partial \dot{q}^j} + \frac{\partial}{\partial \dot{q}^i}\frac{\bar{d}}{dt}\left(\frac{\bar{d}}{dt}\xi^i\right) - \frac{\partial F^j}{\partial \dot{q}^i}\frac{\partial}{\partial \dot{q}^j}\frac{\bar{d}\xi^i}{dt}. \tag{11}$$

It is now enough to consider the divergence with respect to velocities of the symmetry (4) and condition (2) to conclude that

$$\frac{\bar{d}}{dt}I = 0 \tag{12}$$

which is the conservation law. Neither a Lagrangian nor a Hamiltonian is needed in the proof. No previous knowledge of a constant of motion for system (1) is invoked either.

To the best of our knowledge this theorem is unique in the sense that the conservation law is constructed in terms of a symmetry vector of the equations of motion only.

We will now show that this theorem drastically restricts the functional form of point symmetry transformations ζ^i

$$\zeta^i = \zeta^i(q^j, t) \tag{13}$$

for equation (1). Consider the conserved quantity I_0,

$$I_0 = \frac{\partial \zeta^i}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \zeta^i. \tag{14}$$

I_0 reduces to

$$I_0 = 2\frac{\partial \zeta^i}{\partial q^i} \tag{15}$$

due to the fact that ζ^i is velocity independent. I_0 given by (15) is therefore a velocity-independent constant of motion. This fact means that I_0 is a pure number, because there are no dynamical constants which are functions of coordinates only. Then,

$$I_0 = c \tag{16}$$

and, consequently, the most general expression for ζ^i in n dimensions is

$$\zeta^i = \frac{c}{n} q^i + \varepsilon^{ii_2 \ldots i_n} \frac{\partial}{\partial q^{i_2}} (\psi_{i_3 \ldots i_n}) \tag{17}$$

where $\psi_{i_3 \ldots i_n}$ is completely antisymmetric in all of its indices and has arbitrary dependence on q^j and t.

In three dimensions, one has

$$\boldsymbol{\zeta} = \frac{c}{3} \boldsymbol{r} + \nabla \wedge \boldsymbol{\psi} \tag{18}$$

for arbitrary $\boldsymbol{\psi}(\boldsymbol{r}, t)$.

$\boldsymbol{\psi}$ is defined up to the addition of the gradient of an arbitrary gauge function g. Thus, $\boldsymbol{\zeta}$ may be written in terms of two arbitrary functions. Note that the $(n-2)$ form ψ is gauge invariant in the sense that the expression (17) for ζ^i does not change by the addition of the exterior derivative of an arbitrary $(n-3)$-form g_1 to ψ. The $(n-2)$ form ψ (and therefore ζ^i) is also invariant under the addition of the exterior derivative of an $(n-4)$-form g_2 to g_1, so on and so forth. It is not difficult to count the total number of arbitrary functions in ζ^i, which adds up to $(n-1)$, as it should, in view of equations (15) and (16).

The theorem we have presented may now be generalized by relaxing condition (2) in the following fashion. Assume that F^i does not satisfy (2) but there is a function $\lambda = \lambda(q^i)$ such that

$$\frac{\partial F^i}{\partial \dot{q} i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} (\ln \lambda) = 0 \tag{19}$$

(in some coordinate system). Then, the conserved quantity I_λ is

$$I_\lambda = \frac{1}{\lambda} \frac{\partial(\lambda \xi^i)}{\partial q^i} + \frac{1}{\lambda} \frac{\partial}{\partial \dot{q}^i} \left(\lambda \frac{\bar{\mathrm{d}} \xi^i}{\mathrm{d}t} \right). \tag{20}$$

The proof is straightforward and proceeds in the same way as before, by using equations (4), (9) and (10).

As an illustration of the theorem in its general version, consider the case of a two-dimensional harmonic oscillator, in polar coordinates,

$$\ddot{r} = -\omega^2 r + r\dot{\theta}^2 \equiv F^r \tag{21}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \equiv F^\theta. \tag{22}$$

The symmetry equations are

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\,\delta r\right) + \omega^2\delta r - \dot{\theta}^2\delta r - 2r\dot{\theta}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\,\delta\theta = 0 \tag{23}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\,\delta\theta\right) + \frac{2\dot{\theta}}{r}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\delta r) + \frac{2\dot{r}}{r}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\delta\theta) - \frac{2\dot{r}\dot{\theta}}{r^2}\,\delta r = 0. \tag{24}$$

It is easy to verify that

$$\delta r = \varepsilon r^3\dot{\theta} \tag{25}$$

$$\delta\theta = 0 \tag{26}$$

is a particular solution of (23) and (24).

Now, one gets that

$$\frac{\partial F^r}{\partial \dot{r}} + \frac{\partial F^\theta}{\partial \dot{\theta}} = -\frac{2\dot{r}}{r} \tag{27}$$

that is, the force given by (21) and (22) satisfies condition (19) with λ given by

$$\lambda = r^2. \tag{28}$$

In other words, F^i satisfies

$$\frac{\partial F^i}{\partial \dot{q}^i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\ln r^2) = 0. \tag{29}$$

The conserved quantity I_λ is

$$I_\lambda = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(r^3\dot{\theta})) + \frac{1}{r^2}\frac{\partial}{\partial \dot{r}}(r^2(\dot{r}r^2\dot{\theta})) \tag{30}$$

$$I_\lambda = 6r^2\dot{\theta} \tag{31}$$

which is proportional to the angular momentum of the oscillator which is, of course, conserved.

This example shows how the generalized verison of the theorem works and allows for the construction of a conserved quantity without using either Lagrangians or Hamiltonians. Of course, the original version of the theorem is useful for the harmonic oscillator system when the equations are written in cartesian coordinates (using the same symmetry vector given by (25) and (26)).

It is perhaps worth remarking that condition (2) as well as the conserved quantity (6) are not coordinate invariant. Nevertheless, if condition (2) is fulfilled in some particular coordinate system, then the conserved quantity for equation (1) may be constructed by using (6) in that particular coordinate system.

Further applications as well as an extension to field theory will be considered in the future.

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